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# APPLICATION OF PATH SIMULATION TECHNIQUES IN DERIVATIVE PRICING

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**ABSTRACT**
In this paper we consider using path simulation framework for valuation of derivative securities and contracts. We present different derivative pricing methods and put emphasis on their advantages and disadvantages. Special care is taken in presenting use of Monte Carlo methods and how they can be used with path simulations. We have obtained accurate results from simulating Ornstein-Uhlenbeck process with different number of steps and comparing experimental and analytical values for mean and variance. In addition a high precision simulation of generalized Black-Scholes is run for a dividend paying stock to illustrate path simulation methods for analyzing it.
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**Key words:** derivative contracts, derivative pricing, numerical methods, path simulation, Monte Carlo, stochastic processes, binomial lattices, Black-Scholes equation

## BACKGROUND

Derivatives play an important role in modern economy and financial system. It is remarkable that despite recent crisis the total value of the derivatives market (both exchange and privately traded) has not fallen down significantly (Sivy 2013). The significance of derivative securities comes not only from using them as an investment vehicle, but also as they are extremely flexible and effective mean to control and manage risk exposures to different market factors. There is a widespread notion of using the gross value of all derivative contracts as a way to measure the size (and the importance) of the market. Based on different sources and methodologies this value OTC market value ranges from 639 trillion USD (Settlements 2012) to 416.9 trillion USD (ISDA 2012). Although these values sound extremely appealing and attention-raising to the general audience they are not exactly what describes the derivatives market, because for some traded contracts (like for example swaps) it is the market movement that generates value (which by the way can also be negative) and not the nominal value of the outstanding contracts. Therefore a better approach to address general derivative contract market statistic would be by using gros market value (the cost of replacing existing contracts), credit exposure measure (participating parties credit exposure after taking into account for contract netting) and the calculation of contracts netting (measures also used by Bank for International Settlements).

As **Table 1** shows, the official value of these three parameters are far from the general OTC market values, but still represent a vast amount of money.

*Table 1. OTC derivative market information for year 2012 (reported by BIS and ISDA).*

| Condition | Description | |
|---|---|---|
| Total notional amount of outstanding OTC derivatives | 639 trillion USD (BIS) | (-1.0% yr) |
| | 416.9 trillion USD (ISDA) | (-5.3% yr) |
| Gross market value | 25 trillion USD (BIS) | (-7.0% yr) |
| Gross credit exposure | 3.7 trillion USD (BIS) | (-5.1% yr) |

## DERIVATIVE PRICING

There are various methods to address derivative pricing and they all have good and bad sides, which means they are best applied in a limited number of cases, when all the prerequisites are in place. Since it's not always possible to have an overview of all available methods or to come up with their requirements it is better if we are able to define more generic approaches toward solving a pricing problem, instead of defining a lot of special case solutions. Tavella (Tavella 2002) has outlined three basic principles that can support valuation of different derivative contracts:

- lack of arbitrage possibilities would mean that in case market conditions remain the same, a stable price will be achieved for a given derivative contract

Lack of arbitrage possibilities also means that we can use the law of one price and price a contract based on available information for other traded contracts and the assumption that markets are effective (e.g. relative prices should reflect the payoff differences between contracts).

- use of concept for state prices that follows from Arrow-Debreu-McKenzie equilibrium model

This equilibrium model allows us to define so-called Arrow-Debreu security which has only two payoffs – 1 EUR in case of an event happens (or any other kind of condition is fulfilled) and 0 otherwise. The importance of this concept is that if we have a derivative contract which value depends on some underlying asset and is uncertain, then the payoffs of the contract can be modelled with a portfolio of Arrow-Debreu securities. In case this portfolio and the contract have identical payoffs, then they should also have the same present value or the same price (provided that we have no arbitrage possibilities).

- use of expected present value concept

This concept follows from the generic time value of money concept and calculation of probability-weighted payoffs in future. Therefore if we are able to calculate a derivative contract values in future period t, then we could infer that:

$$V_0 = e^{-rt} E_p V_t \quad \text{i.}$$

Where the expected probability-weighted payoffs are denoted by $E_p V_t$ and r is the risk-free rate. It should be noted that probabilities used in $E_p$ are called risk-neutral probabilities and they are different from probabilities of entering a given future state (e.g. objective probabilities).

When used together, those three principles lay the foundation of different methods that let us price derivative contracts. Because it practice derivative contract valuation has to rely on some estimated values (even if they are estimated from available historical data) it is not uncommon to apply more than one method in order to verify the pricing calculations or to improve the final estimates.

We can specify five groups of methods applied to analyze and value a derivative contract:

- valuation based on comparison

This approach can be implemented in a different cases where a comparison can be drawn either between solutions that have already been valuated or between the contract and another set of financial instruments that track its outputs. The main benefit of this approach is that a complex derivative can be analyzed using methods and investment vehicles that are easier to value and interpret. This method is prone to tracking errors when using a combination of financial instruments to mimic the outputs of a derivative contract. These errors may have strong impact and distort the valuation by comparison.

- valuation based on closed-form formulae

A famous application of this approach uses Black-Scholes formula and more generic Black-Scholes equation that describes the price of an option over time (Hull 2008):

$$\frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad \text{ii.}$$

Where he payoff of the option $V(S,t)$ as a function of underlying asset price and time to expiration of the option is known in advance; S denotes the price of the underlying asset; $\sigma$ denotes the variance of the underlying asset; r is the risk-free return.

Solving equation (ii) with different constraints and assumptions on the payoff function, maturity time and underlying asset variance, one can obtain closed form solution for the respective scenarios. The major benefit of this approach is that once a solution of the equation has been found, calculating the value of a specific real option is quite easy. Ease of use, though comes at a price, since relying on Black-Sholes equation suggests that it is possible to have a perfect hedge of the option – which may not always be the case for some real options, especially when they involve assets that are unique (even if they are unique from the point of view/scope of the company). Furthermore real options are typically not traded (and not able to be traded separately) which in addition to not being able to create a replicating portfolio undermines the use of Black-Scholes approach. Last, but not least important many Black-Scholes-like models rely on call-put parity to simplify the mathematical formula derivation. It should be noted that call-put parity (C-P) holds perfectly or very close in case of existing liquid markets and this assumption has to be checked and proven when using the C-P parity for real options.

- valuation through formal transformation of the expectations into partial differential equations

Once expectations have been transformed into partial differential equation, then it is possible to solve it (either formally or with numerical methods) and come up to a solution that, although not a closed-form formulae can help quickly calculate a derivative contract price. One example method of expectation transformation approach is using finite differences to price a derivative. The main advantage of this approach is that it is easier to estimate in advance the computational complexity of the problem and to scale it down in a predictable way (compared to Monte Carlo simulations that we discuss below). Also finite differences method can handle quite well continuous early exercise and discrete sampling of the derivative.

There are also some drawbacks of finite differences use and they are mostly related to meeting the assumptions for Markovian processes and gathering enough data (which may not be trivial in a multi-dimensional model where data grid required by finite difference method may grow quickly).

- valuation based on binomial lattices

Binomial lattices approach, first suggested by Cox, Ross and Rubinstein (Cox, Ross and Rubinstein 1979), can reduce complexity by splitting continuous process into multiple small steps and performing calculations for each step. Although this method is typically used to value options and in particular cases where we need to account for different payments and special cash flow cases, it can be used to value other derivative contracts as well.

- valuation using Monte Carlo simulation

Monte Carlo simulation methods are usable when pricing complex derivatives with complex rules, internal structure and payoff patterns. In such case complexity can be successfully reduced by constructing a large number of value/price paths for the underlying asset and computing derivative payoff for each of them. Then with the available data we can use the third basic principle and discount payoffs to calculate the derivative price. Monte Carlo simulation can be used to analyze different derivative contracts that are of high dimension and high complexity. Use of Monte Carlo analysis in derivative pricing offers some quite useful tools for contract analysis:
Since we're interested in modelling and simulating expectation and expected payoffs the simulation can be oriented either directly toward these payoffs or can be implemented only for estimating probabilities of certain cases.

Simulation model can be designed to account for correlation between different sources of uncertainty/risk (Penev 2008a, Penev 2008b).

Different scenarios can be generated using rules with virtually arbitrary complexity which means that a model can be modified or adjusted toward special cases – like for example early exercise (Milev and Taglaini 2012) or jump diffusion option pricing models where the stock price may experience occasional jumps rather than only continuous changes (Milev 2012). Such models capture the important leptokurtic features of the market better than the standard Black-Scholes model and in contrast to stochastic volatility models are not too complicated so that well-known practical algorithms such as the presented path simulation techniques are easy to implement. The structure of the presented path simulation method makes this approach an extremely flexible interpolation method avoiding most of the frequently met problems in computational finance such as the unstable or slow convergent numerical solutions (Milev and Taglaini 2012), multi-asset valuation (Marchev et. al. 2010, Marchev et. al. 2011), complexity of computer implementation, (see Kabaivanov 2012, Milev, et. al. 2012, Milev et. al. 2013), unreasonable time and memory requirements.

Monte Carlo simulations can be adjusted to avoid (or at least try to reduce the effect of) the so called 'curse of dimensionality' which refers to increasing the computational load with increase in problem dimensions.

## NUMERICAL RESULTS

We demonstrate in this paper a framework for generating different asset paths that can be used to price different derivative contracts. For checking the effectiveness of the generated asset paths we use well known processes and compare the values obtained through simulation with those that can be computed from process characteristics.

**Table 2** demonstrates the behavior of path generation framework with the following processes:

1) Ornstein-Uhlenbeck mean reverting process which is described by equation (iii)

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t \qquad \text{iii.}$$

Where $\theta$, $\mu$ and $\sigma$ are model parameters (subject to the restriction that $\theta$ and $\sigma$ are greater than 0) and $W$ is a Wiener process. In the context of derivative pricing and tracking underlying asset changes; $\mu$ is used to represent mean (or theoretically supported value/price); $\sigma$ is price volatility.

Then $\theta$ indicates how changes in price are absorbed and it reverts to the mean value of $\mu$.

2) Generalized Black-Scholes process which demonstrates a simulation for a process that satisfies equation (ii).

***Table 2.*** *Numerical results for asset path simulation.*

| Process | Description |
|---|---|
| **Ornstein-Uhlenbeck**<br>$x_0 = 0$<br>$\theta = 0.5$<br>$\sigma = 0.3$<br>$\mu = 1$<br>Analytical values:<br>$mean = e^{-\theta t}x_0 + \mu(1 - e^{-\theta t}) = 0.221199$<br>$\text{var} = \frac{\sigma^2 x_0}{2\theta}(1 - e^{-2\theta t}) = 0.0354122$ | When running the simulation we have the following results obtained by generating different number of steps in the process (please refer to row below) |

| Simulations | Mean | Variance |
|---|---|---|
| 100 | 0.202354 | 0.0309014 |
| 1000 | 0.218822 | 0.0353559 |
| 10000 | 0.222322 | 0.0359966 |
| 1000000 | 0.220954 | 0.0354837 |

**Generalized Black-Scholes**
Generalized Black-Scholes simulation is modelled after a dividend paying stock with the following parameters:
Risk free rate of **3.2%** Divident yield of **1.3%** Volatility of **21.4%** Process steps: **200**

**Figure 1** demonstrates how results of the path simulation match theoretically expected ones. As we can see the gap between analytical and simulation values closes very quickly as the number of simulation scenario increases. That makes it possible to produce accurate enough simulations with relatively small number of simulations that do not require extensive computational power.

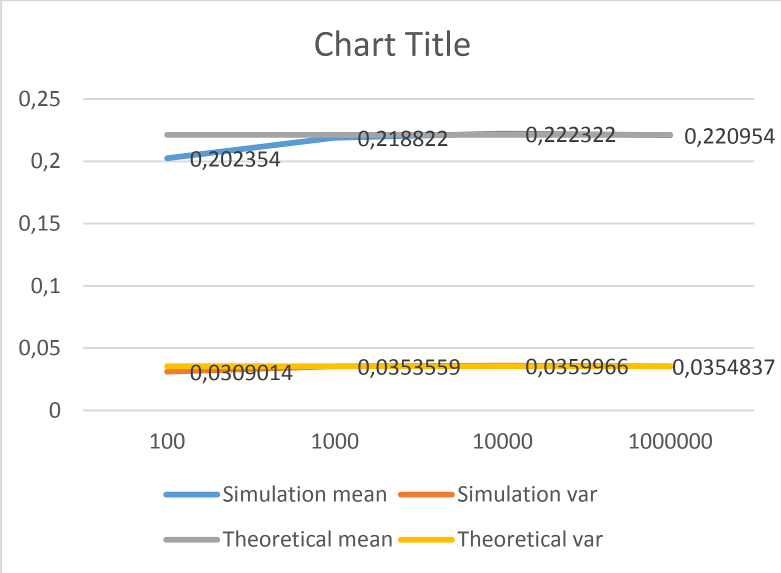


**Figure 1.** Simulation results for Ornstein-Uhlenbeck and convergence with analytical values

**Figure 2** demonstrates evolution of a price under generalized Black-Scholes process, provided that current price is 52 EUR and we have fixed dividend yield for the period of observation. Since simulation can be run with different depth (e.g. different number of scenarios) it can easily be adapted to cover larger periods of time or have higher precision.

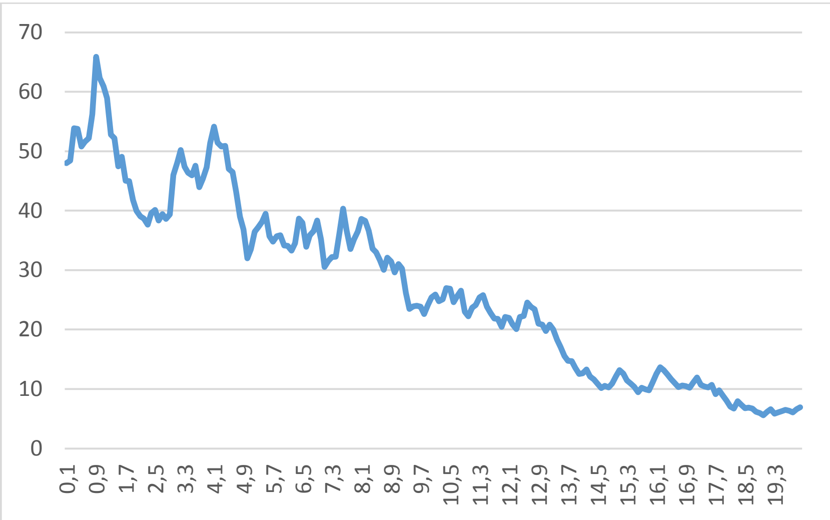


**Figure 2.** High precision simulation of Generalized Black-Scholes proces

## CONCLUSIONS

We have demonstrated how path simulation methods can be used to model complex processes and obtain numeric results with high precision and speed. Although closed form solutions are elegant and easy to understand they are not always applicable to real-life scenarios. There are cases where numeric methods may also turn to be either not adequate enough or too demanding with regard to input data and time for computation. Simulation methods can overcome a lot of obstacles and still provide the tools for derivative pricing with high accuracy and speed. Numerical results accompanying this study show that it is not always necessary to use millions of iterations and simulation scenarios because a good process simulation tends to very closely resemble the analytical values even after relatively small number of simulation scenarios.

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